Damping of nearly inviscid water waves

Carlos Martel[1] and Edgar Knobloch[2]

[1]ETSI Aeronáuticos, Universidad Politécnica de Madrid, 28040 Madrid, Spain
[2]Department of Physics, University of California at Berkeley, Berkeley, California 94720

The complete spectrum of decaying small-amplitude water waves is described. Viscosity is shown to be responsible both for the decay of gravity-capillary waves and for the appearance of a class of viscous modes that are omitted in the standard potential formulation. For sufficiently small viscosity [as measured by the parameter $C \equiv \nu(gh^3)^{-1/2} \ll 1$, where ν is the kinematic viscosity, g the acceleration due to gravity, and h the undisturbed depth of the fluid] the viscous modes decay more slowly than the gravity-capillary ones and must be included in weakly nonlinear theories. The analysis indicates that for realistic values of C second-order corrections to the decay rate of gravity-capillary waves are important and suggests a straightforward resolution of existing discrepancies between experimentally measured and theoretically calculated damping rates. [S1063-651X(97)11311-3]

PACS number(s): 47.20.Ky, 47.54.+r

I. INTRODUCTION

Recent years have seen a resurgence of interest in the Faraday system [1], i.e., the excitation of surface gravity-capillary waves by a time-dependent modulation of the gravitational acceleration. This system exhibits a great variety of pattern-forming behavior [2] and at first sight appears to be relatively simple to analyze. This is because the liquid used in typical experiments, usually water, is almost inviscid [$C \equiv \nu(gh^3)^{-1/2} \ll 1$, where ν is the kinematic viscosity, g the acceleration due to gravity, and h the undisturbed depth of the fluid] and the system admits an elegant Hamiltonian formulation [1]. In fact, as shown below, the small value of C complicates the analysis enormously. This is because in the presence of viscosity there are two types of (unforced) modes: *surface gravity-capillary waves*, which take the form of slowly decaying oscillations, and *viscous* modes, which decay monotonically and are absent in the inviscid case. We refer to the former as inviscid modes since they are present even in the absence of viscosity. Current treatments focus on the former and construct evolution equations for weakly nonlinear gravity-capillary waves, while ignoring the presence of the viscous modes [1,3]. Two types of approaches are used. In the first [1] the Hamiltonian equations (or, equivalently, the equations derived from an averaged Lagrangian) are used to compute the nonlinear terms at third order in the wave amplitude. For waves on a line with reflection symmetry one obtains coupled equations for the complex amplitudes of the modes $\exp(i\omega t \pm ikx)$, where $\omega = \omega(k)$ is the dispersion relation. These equations have purely imaginary coefficients. With viscosity added both modes decay at the same rate. If the inviscid system is parametrically driven the eigenvalues change from being $\pm i\omega$ (twice) to $\pm \lambda_p \pm i\omega$ [$\lambda_p = O(a/g)$, where a is the modulation amplitude], i.e., each eigenvalue of double multiplicity splits, with one of each pair moving into the right half of the complex plane and the other into the left half of the complex plane [4]. The eigenvalues thus form the familiar quartet characteristic of Hamiltonian systems and describe a pair of growing and decaying left- and right-traveling waves. With parametric forc-

ing *and* viscosity, the quartet is shifted to the left by an amount λ_d calculated below. Implicit in this approach is the assumption that the viscous modes decay faster than the driven surface gravity-capillary modes retained in the description. For the applicability of such a description it is critical therefore to identify the slowest decaying modes of the system. As shown below, for $C \ll 1$ the neglected viscous modes typically decay *more slowly* than (or at the same rate as) the surface gravity-capillary waves and hence need to be retained in the theory.

II. DISPERSION RELATION FOR VISCOUS WATER WAVES

The dimensionless equations describing gravity-capillary waves in a layer of viscous incompressible fluid of undisturbed depth h (see Fig. 1) are

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + C\nabla^2 \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0, \tag{1}$$

with the boundary conditions

$$\mathbf{u} = \mathbf{0} \quad \text{at} \quad y = -1 \tag{2}$$

and

$$\zeta_t + u\zeta_x = v, \tag{3}$$

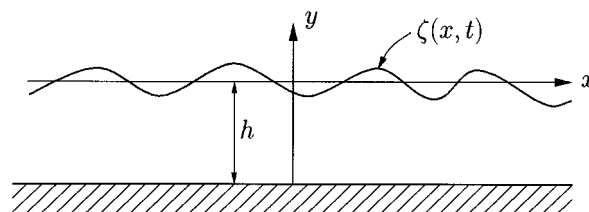

FIG. 1. Sketch of the fluid layer.

$$2C\left(\frac{u_x\zeta_x^2-\zeta_x(u_y+v_x)+v_y}{1+\zeta_x^2}\right)=p-\zeta+T\frac{\partial}{\partial x}\left(\frac{\zeta_x}{\sqrt{1+\zeta_x^2}}\right),$$
$$(4)$$

$$(u_y+v_x)(1-\zeta_x^2)+2\zeta_x(v_y-u_x)=0, \qquad (5)$$

all imposed at the free surface $y=\zeta(x,t)$. All lengths are in units of h and time in units of $\sqrt{h/g}$; the inverse Bond number $T=\sigma(\rho gh^2)^{-1}$ is a dimensionless measure of the importance of surface tension relative to gravity. The solutions of Eqs. (1)–(5), linearized about the basic state $(u,v,p,\zeta)=(0,0,0,0)$, are of the form $(u,v,p,\zeta)=(u_k(y),v_k(y),p_k(y),\zeta_k)\exp(ikx+st)$, where

$$su_k=-ikp_k+C(u_{kyy}-k^2u_k), \qquad (6a)$$

$$sv_k=-p_{ky}+C(v_{kyy}-k^2v_k), \qquad (6b)$$

$$iku_k+v_{ky}=0, \qquad (7)$$

subject to the conditions

$$u_k(-1)=v_k(-1)=0 \qquad (8)$$

and

$$s\zeta_k=v_k(0), \quad 2Cv_{ky}(0)=p_k(0)-(1+k^2T)\zeta_k,$$

$$u_{ky}(0)+ikv_k(0)=0, \qquad (9)$$

together with the requirement $\zeta_0=0$ imposed by mass conservation.

When $k\neq0$ Eqs. (6) and (7) can be written as a single fourth-order eigenvalue problem for $v_k(y)$ whose solution is of the form

$$v_k(y)=a\cosh ky+b\sinh ky+c\cosh qy+d\sinh qy, \quad (10)$$

where $q^2\equiv(s/C)+k^2$. The imposition of the boundary conditions (8) and (9) leads to the dispersion relation [5]

$$k(1+Tk^2)(q\cosh q\sinh k-k\cosh k\sinh q)$$
$$=C^2[4k^2q(k^2+q^2)+(k^4+6k^2q^2+q^4)k\sinh k\sinh q$$
$$-(5k^4+2k^2q^2+q^4)q\cosh k\cosh q], \qquad (11)$$

from which one can compute $q=q(k;C,T)$ and hence obtain $s\equiv C(q^2-k^2)$. For the special case $k=0$ the solutions are of the form $(u_0,v_0,p_0,\zeta_0)\propto(\cos[(n+\tfrac{1}{2})\pi y],0,0,0)$, $n=0,1,2,\ldots$, with $s=-C(n+\tfrac{1}{2})^2\pi^2$, which is always strictly negative.

In the following we study the asymptotic behavior of the solutions s of Eq. (11) when $C\ll1$. We distinguish three different regimes depending on the magnitude of the wave number k.

When $k=O(1)$, i.e., the wavelength of the modes is of the order of the depth of the fluid layer, the dispersion relation (11) has two types of solutions. Earlier work focused on the *gravity-capillary* modes [6]

$$s=\pm i\sqrt{k(1+Tk^2)\tanh k}$$
$$-\frac{[k(1+Tk^2)\tanh k]^{1/4}}{\sinh 2k}k\left(\frac{1\pm i}{\sqrt{2}}\right)\sqrt{C}$$
$$-k^2\left(2+\frac{5+3\tanh^2 k}{16\sinh^2 k}\right)C+\cdots, \qquad (12)$$

characterized by $q_r\sim C^{-1/2}\gg1$ ($q_r>0$). These modes oscillate with $O(1)$ dimensionless frequency; their decay rate is proportional to $\sqrt{C}$, provided that C is sufficiently small. However, the dispersion relation (11) has another class of solutions as well, the *viscous* modes, for which $|q|=O(1)$. At leading order these are given by

$$q\cosh q\sinh k=k\cosh k\sinh q. \qquad (13)$$

This equation has infinitely many solutions of the form $q=i\xi_n(k)$, $n=1,2,\ldots$, where the $\xi_n(k)$ are monotonically decreasing functions of k satisfying

$$\lim_{k\to\infty}\xi_n(k)=n\pi, \quad n\pi\leq\xi_n(k)\leq\left(n+\frac{1}{2}\right)\pi,$$

$$\lim_{k\to0}\xi_n(k)=\xi_n(0), \quad \xi_n(0)=\tan\xi_n(0). \qquad (14)$$

Thus

$$s=-C[k^2+\xi_n^2(k)]+\cdots \qquad (15)$$

and the viscous modes decay *more slowly* than the gravity-capillary modes for sufficiently small values of C. These purely decaying modes are absent from the inviscid formulation and are unaffected by capillary effects at leading order because they do not produce surface deformations. This can be readily seen from Eq. (10), which implies that $|v_k|,|v_{ky}|,|v_{kyy}|=O(1)$ [recall that $k,|q|=O(1)$ for these modes], and Eqs. (6) and (7), which imply $|u_k|=O(1)$ and $|p_k|=O(C)$. From the second condition in Eq. (9) it now follows that $|\zeta_k|=O(C)$.

In the limit of very large wavelengths, $k\ll1$, the solutions of Eq. (11) are given by

$$\widetilde{k}^2(q\cosh q-\sinh q)=-q^5\cosh q, \qquad (16)$$

where $\widetilde{k}=k/C=O(1)$. The frequency $\mathrm{Im}(s)$ of the gravity-capillary modes vanishes for $k\leq C\widetilde{k}_1$, where $\widetilde{k}_1=1.3650$. At this point these modes split into two real decaying modes with $\mathrm{Re}s(k)\to0$ and $\mathrm{Re}s(k)\to-(\pi/2)^2C$, as $k\to0$. The viscous modes are also given by Eq. (16), but with $q=iq_i$. For these modes $s(k)\to-C(n+\tfrac{1}{2})^2\pi^2$, $n=1,2,\ldots$, as $k\to0$, and the viscous modes approach the $k=0$ results, except that $s(0)=0$ is excluded by mass conservation.

When $k\gg1$ the viscous modes [$q=O(1)$] decay at the rate $s=-C(k^2+n^2\pi^2)$, while the gravity-capillary modes ($q_r\gg1$) satisfy

$$\widetilde{q}^4+2\widetilde{q}^2\widetilde{k}^2-4\widetilde{q}\widetilde{k}^3+\widetilde{k}^3+\widetilde{k}^4=0,$$

$$\widetilde{k}\equiv kC^2/T, \quad \widetilde{q}\equiv qC^2/T, \qquad (17)$$
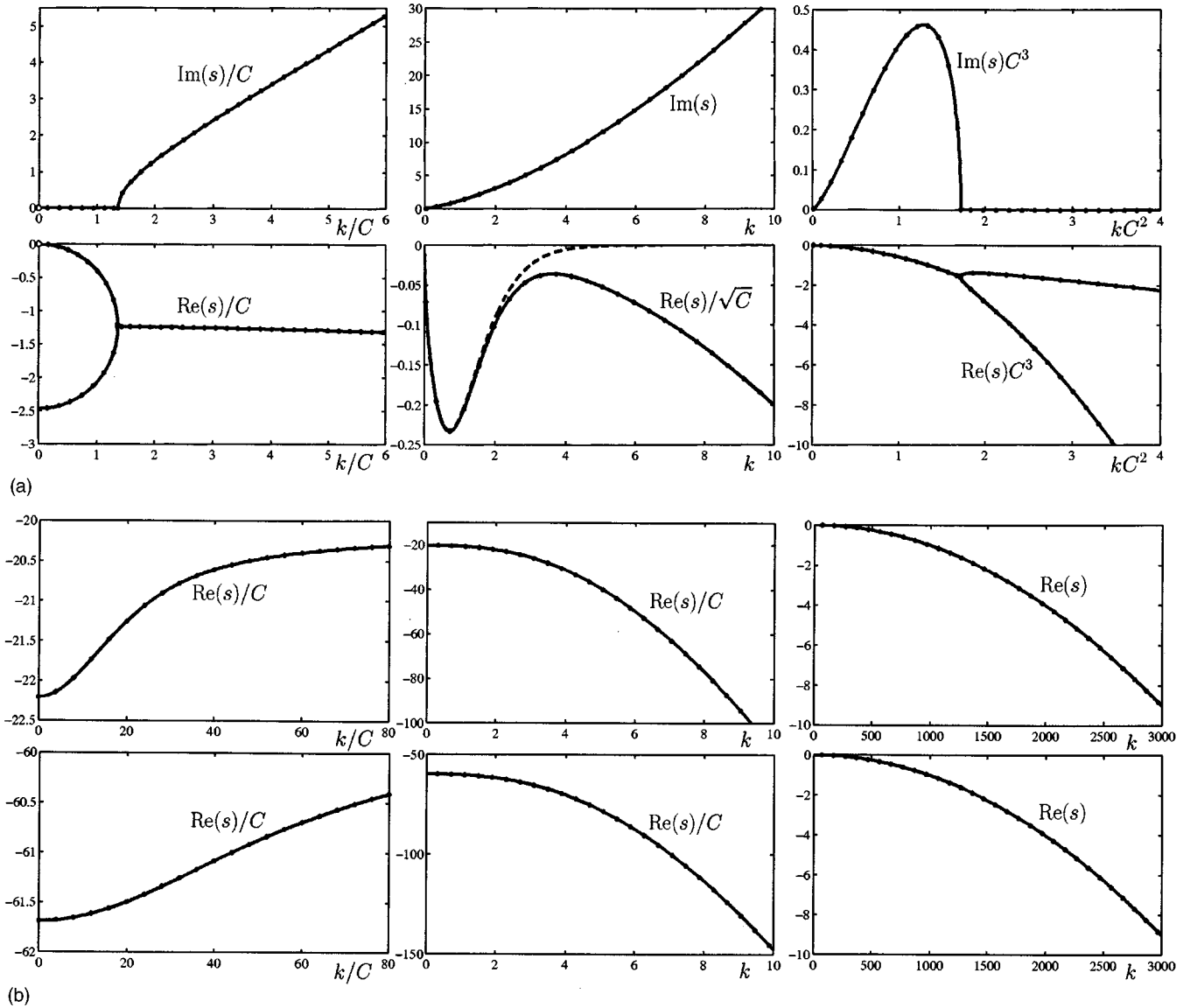
FIG. 2. Dispersion relation $s(k)$ for $T=1$ and $C=10^{-6}$ in the three ranges of k identified in Sec. II: (a) gravity-capillary ("inviscid") modes and (b) first two viscous modes. Dots indicate the exact solution of Eq. (11); the asymptotic results (12) through $O(C^{1/2})$ and $O(C)$ are indicated by dashed and solid lines, respectively.

with $\widetilde{k}=O(1)$ and $\widetilde{q}=O(1)$. For $k \gtrsim \widetilde{k}_2 T/C^2$, where $\widetilde{k}_2=1.7200$, the capillary modes are overwhelmed by viscosity and again split into a pair of decaying real modes whose leading-order decay rates are given by

$$s=-0.9124Ck^2+\cdots, \quad s=-\frac{T}{2C}+\cdots. \quad (18)$$

These results are illustrated in Fig. 2 for $C=10^{-6}$ and $T=1$. The figures show (a) $\mathrm{Re}(s)$ and $\mathrm{Im}(s)$ for the slowest-decaying gravity-capillary mode and (b) $\mathrm{Re}(s)$ for the two slowest-decaying viscous modes, as functions of k for the three different ranges of k identified above. The dots indicate the exact solutions of the dispersion relation (11) computed using numerical continuation techniques; the dashed lines show the decay rate $\mathrm{Re}(s)$ given by Eq. (12) truncated at $O(C^{1/2})$ while the solid lines show the results from the truncation at $O(C)$. Evidently, for $k\approx 2$ or greater the second

term in Eq. (12) must be retained. As discussed further below, this effect becomes more important for larger (and more realistic) values of $C\ll 1$, i.e., the leading-order asymptotic approximation to the decay rate provides a poor approximation to the decay rate under experimentally relevant conditions. This is because the $O(C^{1/2})$ decay rate of the gravity-capillary modes decreases exponentially with increasing k and eventually becomes negligible compared with the $O(C)$ term. This is a consequence of the fact that the damping that comes from the boundary layer at the bottom wall becomes exponentially small as k increases, allowing the $O(Ck^2)$ viscous dissipation in the body of the fluid to dominate.

III. DISCUSSION

As mentioned in Sec. I, current treatments of the Faraday instability ignore the presence of the viscous modes and hence are formally valid only when
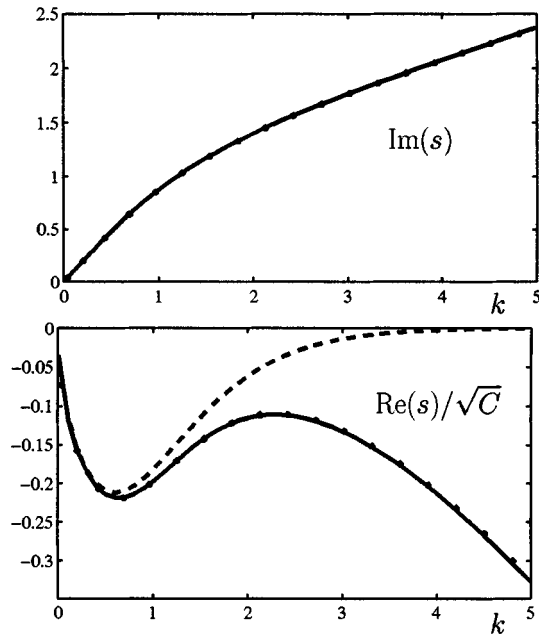
FIG. 3. Gravity-capillary modes for $k=O(1)$, plotted as in Fig. 2, but for the parameter values used in [7]: $T=0.51\times10^{-2}$ and $C=0.43\times10^{-4}$.

$$\frac{a}{g}+\frac{\lambda_d}{\omega}\ll\frac{Ck^2}{\omega},\qquad(19)$$

where, for $k=O(1)$, $-\lambda_d$ is given by the real part of Eq. (12). Since the viscous modes decay more slowly than the (unforced) gravity-capillary modes when C is less than $O(10^{-4})$, this condition cannot be satisfied in small viscosity liquids. In fact, we have examined a large number of the experiments described in the literature and find that a/g is always larger than 0.05, i.e., much larger than the $O(C)$ magnitude implied by Eq. (19).

These results indicate that the current theoretical understanding of nominally inviscid experiments may require reexamination. This conclusion is supported by the disagreement between predicted and observed decay rates of the first few natural modes of oscillation [7,8]. This disagreement persists even when steps are taken to eliminate uncertainties due to dissipation in the meniscus, at the free surface, or in corners of the experimental apparatus, as in the experiments of Henderson and Miles [7]. In these experiments a circular cylinder is filled to the brim, pinning the contact line to the (sharp) rim of the container. The circular nature of the container reduces the unknown dissipation that might arise as a result of the presence of corners. The authors measured the decay rates of the first few natural (unforced) oscillation modes in such a configuration for both clean and contaminated surfaces. While the measured frequencies agreed well with leading-order asymptotics, the decay rates for a clean surface differed from the predicted ones by as much as 300%. In Fig. 3 we show the frequencies and decay rates of gravity-capillary waves in a horizontally unbounded layer for the parameter values employed in the experiments of Henderson and Miles [7], i.e., $T=0.51\times10^{-2}$ and $C=0.43\times10^{-4}$. The results from leading-order asymptotics (12) through $O(C^{1/2})$ and $O(C)$ are indicated by dashed and

solid lines, respectively, and compared with the exact result (points) computed directly from Eq. (11). Although the frequencies match very accurately the leading-order asymptotic prediction, this is not so for the decay rates. For $k>1$ the leading-order asymptotic prediction underestimates the decay rate by an $O(1)$ amount. However, the figure also shows that, with the $O(C)$ correction retained in Eq. (12), the asymptotic prediction reproduces the exact result very well. Indeed, the effect of this correction is of the right order of magnitude to suggest that the discrepancy between the measured and predicted decay rates in the Henderson-Miles experiments will disappear if the theoretical predictions include $O(C)$ corrections to the decay rate. This suggestion will be explored elsewhere [9] and is supported by the observation that the discrepancy between theory and measurement increases with the wave number; cf. Fig. 3. A similar observation resolves discrepancies between leading-order asymptotics and measured decay rates of oscillations of liquid bridges, as discussed by Higuera et al. [10].

For the values of C used by Henderson and Miles [7] the damping rates of the gravity-capillary and viscous modes are comparable. Despite this, the measured damping rates of the gravity-capillary modes are most likely not contaminated by the presence of the viscous modes because the damping rates were obtained through surface elevation measurements and, at leading order, the viscous modes do not deform the free surface.

As mentioned in the Introduction, a detailed knowledge of the nearly neutral modes is of paramount importance in the development of a weakly nonlinear theory for the Faraday system. The analysis presented in this paper demonstrates that in typical experiments [7,8] the gravity-capillary and viscous modes decay with similar decay rates. Consequently, the appropriate normal form equations governing weakly nonlinear evolution of the driven modes must take into account the infinitely many branches of almost neutral viscous modes. These manifest themselves at second order in the wave amplitude as viscosity-induced slowly varying streaming flows [11] and these in turn contribute to the coefficients of the cubic terms in the amplitude equations for the gravity-capillary modes [12]. These effects are known to be important in the theory of (unforced) water waves [13] and cannot be captured by a theory based on the potential formulation: the small viscosity limit is a singular limit that cannot be successfully treated as a regular perturbation of the inviscid case. Similar conclusions apply to nearly inviscid liquid bridges, as discussed in detail by Nicolás and Vega [14].

[1] J. Miles and D. Henderson, Annu. Rev. Fluid Mech. **22**, 143 (1990).

[2] A. Kudrolli and J.P. Gollub, Physica D **97**, 133 (1996).

[3] A.B. Ezerskiĭ, M.I. Rabinovich, V.P. Reutov, and I.M. Starobinets, Sov. Phys. JETP **41**, 157 (1986); S. Douady, S. Fauve, and O. Thual, Europhys. Lett. **10**, 309 (1989); S. Douady, J. Fluid Mech. **221**, 383 (1990); A.D.D. Craik and J.G.M. Armitage, Fluid Dyn. Res. **15**, 129 (1995).

[4] E. Knobloch, A. Mahalov, and J.E. Marsden, Physica D **73**, 49 (1994).

[5] K. Kumar, Proc. R. Soc. London, Ser. A **452**, 1113 (1996); A.B. Basset, *A Treatise on Hydrodynamics* (Dover, New York, 1961), Vol. 2 gives this result for $T=0$.

[6] A similar expansion arises in the theory of viscous cross waves: see A.J. Bernoff, L.P. Kwok, and S. Lichter, Phys. Fluids A **1**, 678 (1989).

[7] D.M. Henderson and J.W. Miles, J. Fluid Mech. **275**, 285 (1994).

[8] D.M. Henderson and J.W. Miles, J. Fluid Mech. **213**, 95 (1990); L. Jiang, C.-L. Ting, M. Perlin, and W.W. Schultz, *ibid.* **329**, 275 (1996).

[9] C. Martel, J.A. Nicolás, and J.M. Vega (unpublished).

[10] M. Higuera, J.A. Nicolás, and J.M. Vega, Phys. Fluids **6**, 438 (1994); M. Higuera and J.A. Nicolás, *ibid.* **9**, 276 (1997).

[11] H. Schlichting, Z. Phys. **33**, 327 (1932); M.S. Longuet-Higgins, Philos. Trans. R. Soc. London, Ser. A **245**, 535 (1953).

[12] C. Martel, J.M. Vega, and E. Knobloch (unpublished).

[13] For a review see A.D.D. Craik, J. Fluid Mech. **116**, 187 (1982).

[14] J.A. Nicolás and J.M. Vega, J. Fluid Mech. **328**, 95 (1996); J.A. Nicolás, D. Rivas, and J.M. Vega, Z. Angew. Math. Phys. **48**, 389 (1997).